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Energy-Momentum Properties and the Spin of Short-Lived Particles[1]

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The space–time translation property of a stable particle is characterized by a time-like Lorentz vector $(E, \vec{k})$. We show in this contribution that unstable particles are, in addition, characterized by a space-like Lorentz 4-vector of uncertainties, or spreads, $(\Delta E, \vec{\Delta k})$. This is true for unstable states created in formation-, in production-, and in decay-experiments. The space-like nature of the spread vector causes a nonzero momentum spread to be present in all Lorentz frames so that there is no Lorentz frame in which the unstable particle is entirely at rest. With the space-like spreadvector $(\Delta E, \vec{\Delta k})$ in addition to the time-like $(E, \vec{k})$, also the rotation property of an unstable particle is affected, and unstable states have an uncertainty in their spin. This means neighboring spin states are occupied in addition to the original spin. Experiments are discussed that show a principal limitation of the accuracy of spin measurement from finite lifetimes. Wave functions for unstable particles are discussed, and we show in the example of a short-lived spin-0 state that the appearance of a spin neighbor in the amplitude is proportional to the inverse lifetime.

KEY WORDS: unstable particle; momentum speed; spin spread; spin uncertainty; space-like spread vector; spin neighbors; complex energy momentum.

1. INTRODUCTION

If one tries to describe stable and unstable particles with the same concepts, then the instability would appear as an additional property of a particle, and a stable one would represent a limiting case of an unstable one. This viewpoint is the more desirable as stable and unstable states have often a similar behavior like sharing the same SU(3)-multiplet or having comparable production cross sections.

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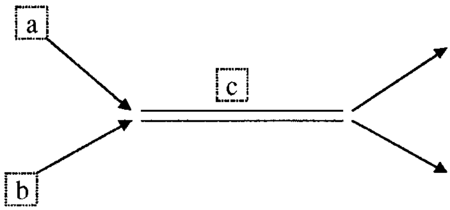

Fig. 1. Formation of a resonance c between particles a and b by external variation of the energy of (a, b).

Following this line, we do not define resonances[4] as scattering processes between stable particles, and we do not wish to refer to any *dynamic* property of the unstable states or their decay products. In a previous paper (Blum and Saller, 2003) we have constructed relativistic wave functions in space and time of unstable states which depend only on the Lorentz properties of the unstable state itself.

In this contribution we would like to emphasize this purely kinematic description of unstable states and show that their decay in time implies an uncertainty of momentum and of spin. We hope to make it clear that it is the Lorentz group which produces these effects—one might say that the spread ΔE in energy of a state decaying in time cannot be separated from the other Lorentz parameters and brings about a spread Δk in momentum and a spread ΔS in the spin of the decaying particle.

2. ENERGY–MOMENTUM PROPERTIES OF UNSTABLE STATES

Unstable states or resonances come into existence either "in formation" or "in production" or "in a decay" of a more massive particle.

2.1. Unstable States Created in Formation

The resonance occurs as an intermediate state in the scattering of two stable particles a, b (Fig. 1).

$$a + b \to c \to a + b$$

Experimentally, the scattering energy is fixed for a number of events and then systematically varied in steps. At every energy step the events are recorded and the cross section is calculated.

[4] In this paper we use the terms "unstable state," "unstable particle," "resonance," or "short-lived particle" as having the same meaning, and we use the term "stable particle" for a particle where the inverse lifetime is zero or can be neglected in the given context.

Typically the cross section σ goes through a maximum indicating the presence of the resonance. The ensemble of all recorded scattering events represents this unstable state. A well-known example is the scattering on hydrogen nuclei of π^+ with laboratory momenta between 200 and 600 MeV/c peaking at 390 MeV/c and thus indicating the presence of the $N^{*++}(1230)$ unstable state.

The energy–momentum properties of such a state are characterized first by (E, k) at the peak, where k is the same as the momentum of the incoming particle a at σ_{peak} and

$$E = \sqrt{k^2 + m_a^2} + m_b,$$

and second by the width of the resonance, for example the momentum difference Δk where the cross section is half the peak value. The corresponding energy difference is given by

$$\Delta E = \left(\sqrt{(k + \Delta k)^2 + m_a^2} + m_b\right) - \left(\sqrt{k^2 + m_a^2} + m_b\right). \tag{1}$$

It is important to note that

$$\Delta E \leq \Delta k \tag{2}$$

which can be seen when developing Eq. (1) w.r.t. the small quantity $2k\Delta k/(k^2 + m_a^2)$ and neglecting terms $(\Delta k)^2$. The equal sign in Eq. (2) occurs for $m_a = 0$. The property (2) is obviously independent of the particular choice of the definition of the width or the central value. The definition of Δk and ΔE given here is only up to a factor and will be completed in subsection 2.4.

Under Lorentz transformations, (E, k) actually represents a 4-vector and should be written $(E, \vec{k})$. The same is true for $(\Delta E, \Delta k) \to (\Delta E, \overrightarrow{\Delta k})$. Equation (2) actually refers to the magnitude of $\overrightarrow{\Delta k}$ and is

$$\Delta E \leq |\overrightarrow{\Delta k}|. \tag{3}$$

This is seen by repeating the steps from (1) to (2) using vectors $\vec{k}$ and $\overrightarrow{\Delta k}$. In the laboratory system $\overrightarrow{\Delta k}$ is parallel to $\vec{k}$, but not in every Lorentz system.

2.2. Unstable States Created in Production

The unstable state is made as one entity in a collision involving several particles, and then propagates force-free through space and time. Eventually it decays according to a law of radioactivity. An example is the W particle as it was produced in pp collision when it was discovered (Fig. 2).

$$a + b \to c + d + e + \cdots (c \to x + y + z + \cdots) \tag{4}$$

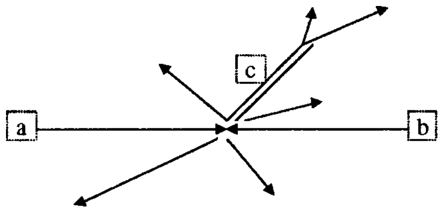

Fig. 2. Production of a free unstable particle or resonance c in coproduction with other particles.

In terms of the kinematic variables of particle c we may describe the most general condition of its birth by the two-body reaction

$$a + b \to c + d \ \ (c \to x + y + z + \cdots) \tag{5}$$

where d represents the energy–momentum sum of all the other particles produced in the same reaction and assumed to be stable. After specifying the total energy $\sqrt{s}$, the masses m_c and m_d we may calculate in the center of mass system of reaction (5) the energy E_c and momentum k_c of particle c. The conservation laws determine them to be

$$\begin{aligned} E_c &= \sqrt{s}(1 + u - v)/2 \\ k_c &= \sqrt{s}\sqrt{(1 + u - v)^2 - 4u}/2, \end{aligned} \tag{6}$$

with the abbreviations $u = m_c^2/s$ and $v = m_d^2/s$. If the mass m_c does not have a sharp value but is statistically distributed around its central value m_c with variation $\pm\Gamma/2$, the ensuing variations in E_c and k_c are in first-order of $\Gamma/2$,[5]

$$\begin{aligned} \Delta E &= \frac{\partial E}{\partial u}\frac{du}{dm}\Gamma/2 = \sqrt{u}\Gamma/2 \\ \Delta k &= \frac{\partial k}{\partial u}\frac{du}{dm}\Gamma/2 = \frac{-1 + u - v}{\sqrt{(1 + u - v)^2 - 4u}}\sqrt{u}\Gamma/2 \end{aligned} \tag{7}$$

We note that the ratio $\Delta E/\Delta k$ is always in the interval

$$-1 \leq \Delta E/\Delta k < 0 \tag{8}$$

because the domain of u and v is restricted to $0 < u < 1$, $0 \leq v < 1$, $|\sqrt{u}| + |\sqrt{v}| < 1$.

Again, under Lorentz transformations (E, k) behaves as a 4-vector and should be written $(E, \vec{k})$, and the same is true for $(\Delta E, \vec{\Delta k})$. In the center of mass system of reaction (5) $\vec{\Delta k}$ is parallel to $\vec{k}$, but not in every Lorentz frame.[6] Just as in Eq. (3)

[5] The index c is omitted from here onwards.

[6] More detailed arguments are presented in (Blum and Saller, 2003).

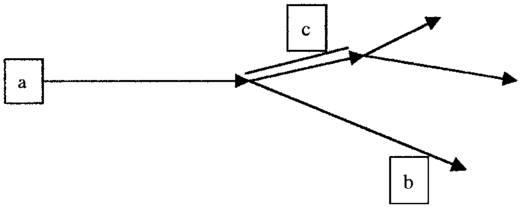

Fig. 3. Production of a free unstable particle or resonance c in the decay of a more massive particle a.

we have for the unstable state in production

$$\Delta E \leq |\Delta \vec{k}|. \tag{9}$$

2.3. Unstable States Created in Decay

The unstable state is created as one entity in the decay of a more massive particle which is assumed to be metastable, i.e. its own natural mass width, being nonzero because it decays spontaneously, can be neglected (Fig. 3).

$$a \to b + c \ (c \to x + y + z + \cdots) \tag{10}$$

Particle b represents the energy–momentum sum of all the other particles that are produced in the same decay. They are assumed to be stable.

This creation of the unstable state in the decay of a more massive particle is a special case of the creation "in production" treated in the previous subsection. Identifying $\sqrt{s}$ with the mass of the decaying a, and particle d in Eq. (5) with particle b in Eq. (10) we recuperate Eqs. (6) to (8). In particular we note that also in the "creation in decay" the widths of energy and momentum of the unstable state are related by

$$\Delta E \leq |\Delta \vec{k}|. \tag{11}$$

2.4. Common Properties of Unstable States

Independent of the way in which they were created, unstable states are characterized by an energy-momentum "spread vector" $(\Delta E, \Delta \vec{k})$ which is space-like because of the relations (3), (9), and (11). The time-like $(E, \vec{k})$ and the space-like $(\Delta E, \Delta \vec{k})$ give rise to two types of Lorentz frames which we give the following names:

$$\text{Central rest system: } \vec{k} = 0 \tag{12}$$

$$\text{Sharp-energy system: } \Delta E = 0 \tag{13}$$

The space-like property holds in all Lorentz frames, especially in the central rest frame of the unstable state (variables denoted by *). It is the same Lorentz boost which transforms

$$(E, \vec{k}) \to (m, 0) \quad \text{and} \quad (\Delta E, \vec{\Delta k}) \to (\Delta E^*, \vec{\Delta k}^*) = (\Gamma/2, \vec{\Delta k}^*)$$

We may identify ΔE^* with the width $\Gamma/2$ of the mass distribution. (This also completes the definition of ΔE and Δk in subsection 2.1 which was only given up to a factor). The space-like nature of the spread vector requires

$$|\vec{\Delta k}^*| \geq \Gamma/2 \tag{14}$$

There is an unavoidable momentum uncertainty in the central rest frame of the unstable particle.

One may visualize the energy-momentum spread vector as a correlation between the deviations in energy and momentum from their central values $(E, \vec{k})$. A well-defined unstable state consists of an ensemble grouped around a central $(E, \vec{k})$. When measured in an experiment, this state produces many events $(E_i, \vec{k}_i)$ which follow a (highly specialized four-dimensional) probability distribution around $(E, \vec{k})$—just like a stable state in a given angular momentum produces a distribution function of the corresponding angles. The correlation between the individual deviations $E_i - E, \vec{k}_i - \vec{k}$ may be written in the following form:

$$\frac{E_i - E}{\Delta E} = \frac{k_i^x - k^x}{\Delta k^x} = \frac{k_i^y - k^y}{\Delta k^y} = \frac{k_i^z - k^z}{\Delta k^z} \quad \text{for all} \quad i \tag{15}$$

We write $\vec{k} = (k^x, k^y, k^z)$ etc., and it is understood that in Eq. (15) the deviations in the numerators are zero when the corresponding denominator vanishes.

This suggests a graphical representation of unstable states as in Fig. 4. In a diagram E vs. $k \equiv |\vec{k}|$ we draw three hyperbolas for a given mass m and width $\pm\Gamma/2$, satisfying $E^2 - k^2 = (m - \Gamma/2)^2, m^2$ and $(m + \Gamma/2)^2$. The events belonging to such a state at a given E and k are points on a straight line intersecting the hyperbola m at E, k under some angle. We take this line to represent the unstable state; being space-like, it can never be steeper than 45° (given equal scales of E and k).

The two 4-vectors have three invariants between them

$$\begin{aligned} (E, \vec{k})^2 &= m^2 > 0 \\ (E, \vec{k}) \cdot (\Delta E, \vec{\Delta k}) &= m\Gamma/2 \\ (\Delta E, \vec{\Delta k})^2 &= -B^2 < 0 \end{aligned} \tag{16}$$

The third invariant we have called B^2 (capital beta). The second one is best evaluated in the central rest frame.

The most general unstable state is thus characterized by 8 real numbers. These can also be chosen as: 5 to specify the unstable state in its central rest frame, and

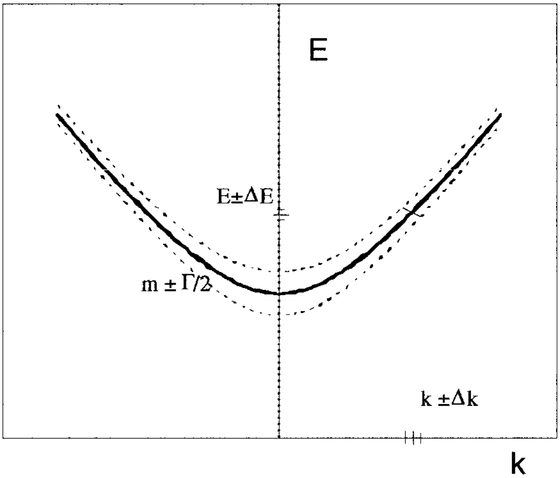

Fig. 4. Representation of an unstable particle in the energy–momentum plane by a straight line through the point (E, k) on the hyperbola (m). The relation between ΔE, Δk and Γ is displayed.

3 for the Lorentz boost:

$$\begin{pmatrix} m \\ 0 \end{pmatrix}, \begin{pmatrix} \Gamma/2 \\ \Delta\vec{k}^* \end{pmatrix} \xrightarrow{\vec{\beta}} \begin{pmatrix} E \\ \vec{k} \end{pmatrix}, \begin{pmatrix} \Delta E \\ \Delta\vec{k} \end{pmatrix} \tag{17}$$

A special case is given with six parameters when $\Delta\vec{k}$ is parallel to $\vec{k}$, and therefore to $\vec{\beta}$ in Eq. (17). If we further fix the value of $|\vec{\beta}|$ such that we arrive in the sharp-energy system (13), only five parameters are left.

Traditionally (Bell and Steinberger, 1966; Bohm, 1993; Bohm and Kaldass, 1999) one specified five parameters for the most general case, for example the width of the rest mass distribution in addition to the central energy-momentum $(E, \vec{k})$ or, equivalently, two for the rest frame and three for the Lorentz boost:

$$\begin{pmatrix} m \\ 0 \end{pmatrix}, \begin{pmatrix} \Gamma/2 \\ 0 \end{pmatrix} \xrightarrow{\vec{\beta}} \begin{pmatrix} E \\ \vec{k} \end{pmatrix}, \begin{pmatrix} \Delta E \\ \Delta\vec{k} \end{pmatrix} \tag{18}$$

This results in a time-like spread vector, incompatible, as shown above, with energy-momentum conservation in the creation of the unstable state.

The two characterizations of resonances have a completely different behavior under Lorentz transformations as illustrated in the two Figs. 5 and 6 and their captions. The essential element is the nonzero $\Delta\vec{k}^*$, the momentum uncertainty in the central rest system. Of all the events in the mass distribution, only the center part is at rest,[7] the events in the high and low wings actually fly away in opposite

[7] Hence the name given to the condition (12).

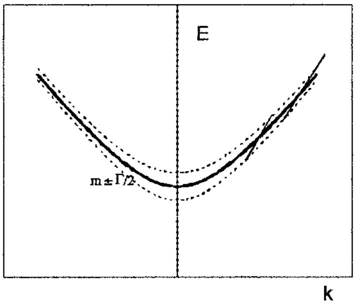

Fig. 5. A resonance at three momenta k in the traditional description. The lines to represent the resonance are constrained to point to the origin, the energy spread ΔE remains proportional to E, and Δk remains proportional to k. In the rest system, the momentum spread Δk is zero, and the only spread is in the mass, namely $\pm\Gamma/2$. Once m, Γ, k are given, the line which represents the resonance is fixed.

directions with velocities of the order of $\pm\Delta\vec{k}^*/m$ which is at least as large as $\pm(\Gamma/2)/m$. There is no Lorentz system in which the unstable state comes entirely to rest.

2.5. Complex Energy–Momentum

In (Blum and Saller, 2003) we have constructed relativistic wave functions in space and time for states characterized by $(E, \vec{k})$ and $(\Delta E, \Delta\vec{k})$. They were obtained from the harmonic expansion (Fourier expansion) of the Feynman propagator, taking into account the correlations indicated by Eq. (15). A plane wave is represented by

$$\psi(t, \vec{x}) = e^{-i(E-i\Delta E)t}\, e^{i(\vec{k}-i\,\Delta\vec{k})\cdot\vec{x}}, \tag{19}$$

a spherical wave where both $\Delta\vec{k}$ and $\vec{k}$ are directed radially outward, by

$$\psi(t, r) = e^{-i(E-i\Delta E)t}\, \frac{1}{r}\, e^{i(k-i\Delta k)r}. \tag{20}$$

These wave functions hold in first-order of Γ/m and are not yet normalized. See Section 3.3 further down for more details.

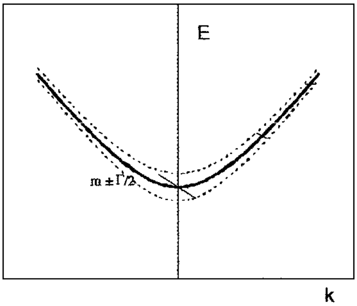

Fig. 6. A resonance created in production at three momenta k in our description. The lines to represent the resonance exhibit a slope which cannot be larger than 1 in magnitude. Once m, Γ, k are given, the slope of the line which represents the resonance is not yet fixed but one more element from the production process has to be known. In the central rest system, the energy spread is $\pm\Gamma/2$, and the momentum spread $\mp\Delta k^*$ is at least as large. The element from production determines how much larger it is.

As an argument in Eq. (19) the energy-momentum spread appears as the imaginary part of a complex 4-vector comprising all 8 real values. In lieu of Eq. (17) one may also write

$$\begin{pmatrix} m - i\Gamma/2 \\ -i\Delta\vec{k}^* \end{pmatrix} \xrightarrow{\vec{\beta}} \begin{pmatrix} E - i\Delta E \\ \vec{k} - i\Delta\vec{k} \end{pmatrix} \tag{21}$$

The boost parameter is still the real 3-vector it was before although the unstable particle "travels with a complex velocity" which means that the three complex momentum components, divided by the complex energy component, result in a "velocity" that cannot be real.

3. THE SPIN OF AN UNSTABLE STATE

The situation in the central rest frame, as it was described in the paragraph of Section 2.4 belonging to Eq. (14), creates complications for the definition of spin: The intrinsic spin of the field in question is modified by the presence of additional angular momentum created by the momentum uncertainty Δk^* in the occupied space. Before we proceed to a more formal treatment we wish to present four direct arguments why the angular momentum of any short-lived state cannot be infinitely sharp but acquires an uncertainty which we call "angular momentum spread" or "spin spread."

3.1. Four Direct Arguments in Favor of an Angular Momentum Spread of Short-Lived Particles

3.1.1. The Spin of a Short-Lived State Can Only be Measured With a Limited Accuracy

Imagine an experiment in which the amount J of spin angular momentum of a short-lived state is measured using the method of nuclear spin resonance in a magnetic field B. The precession frequency ω which is observed for nonzero spin determines J, if the magnetic moment μ is known (Fig. 7), then

$$J = \frac{\mu B}{\omega}.$$

Let the excited state have energy E, above the ground level E_0, width Γ and mean life $\tau = \hbar/\Gamma$. Now the frequency can only be measured during the lifetime of the state and is therefore limited to the accuracy $\Delta\omega \approx 1/\tau$, so that the accuracy of the measurement of J is given by

$$\Delta J \approx \frac{\Delta\omega}{\omega^2}\mu B \approx \frac{1}{\tau}\frac{J^2}{\mu B},$$

which becomes smallest for the largest value of μB, and a small J.

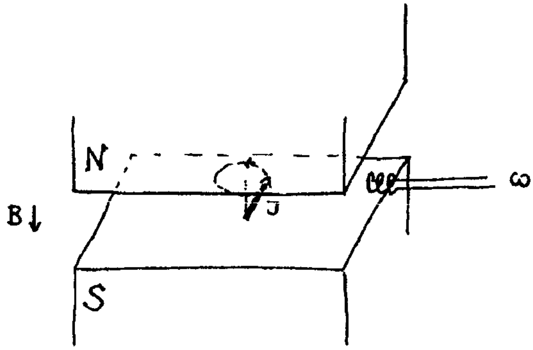

Fig. 7. Principle of a nuclear spin measurement by extraction of the precession frequency.

In this experiment the interaction energy between the apparatus and the excited state is represented by μB; it cannot be made as large as the value of $E - E_0$ if the state to be measured should remain distinguished from the ground state. Therefore we have to keep $\mu B < E - E_0$. Assigning the lowest integer non-zero value to J, $J = \hbar$, we arrive at an uncertainty that cannot be smaller than

$$\Delta J \approx \frac{1}{\tau}\frac{\hbar^2}{E - E_0} = \frac{\hbar\Gamma}{E - E_0}$$
$$\frac{\Delta J}{\hbar} \approx \frac{\Gamma}{E - E_0} \tag{22}$$

3.1.2. The Angular Momentum Component of a Short-Lived State Can Only be Measured With a Limited Accuracy

In a Stern–Gerlach experiment (Fig. 8) let a beam of short-lived states of energy E be created from a beam of atoms that are in the ground state with energy

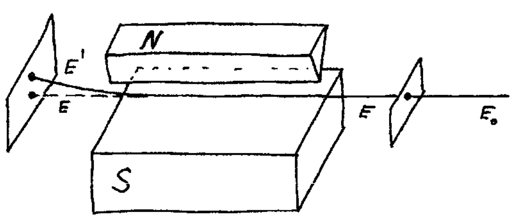

Fig. 8. Principle of a Stern–Gerlach experiment for the measurement of the spin component in the direction of the field inhomogeneity.

E_0. In the magnetic field B the presence of an angular momentum component $m\hbar$ in the direction of the field inhomogeneity changes the energy to be $E' = E - \mu B$, where μ is the magnetic moment of the state. Knowing μ, one measures m by recording the energy difference; then m is given by

$$m = \frac{E' - E}{\mu B}.$$

The energy difference $E' - E$ is determined from a measurement of the displacement of the beam in the inhomogeneous field.

It is well known that a minimal length of time, T, is required to measure the energy difference $E' - E$; for shorter times the beam would still overlap the reference beam.

$$T > \frac{\hbar}{E' - E}.$$

The unstable state can only be observed during its lifetime; on the average we must involve the mean lifetime τ. Therefore there is a limit to the accuracy $\Delta(E - E')$ with which the value of $E' - E$ can be determined:

$$\Delta(E' - E) > \frac{\hbar}{\tau}$$

$$\Delta m = \frac{\Delta(E - E')}{\mu B} > \frac{\hbar}{\tau \mu B}.$$

The accuracy Δm becomes better if μB is increased. But again, B cannot be made arbitrarily large. If μB were made as large as the energy difference $E - E_0$, the experiment would lose its meaning as the state E would no longer be distinguished from E_0. Therefore, the uncertainty Δm with which the angular momentum component can be measured has a lower limit

$$\Delta m > \frac{\hbar}{\tau(E - E_0)} = \frac{\Gamma}{E - E_0}, \tag{23}$$

where $\Gamma = \hbar/\tau$ is the width of the unstable state.

3.1.3. A Short-Lived State Cannot Have a Sharp Orbital Angular Momentum

In a two-body-bound state the angle ϕ of rotational motion, and the angular momentum L of the state are two variables that are conjugate to each other, and the uncertainty principle requires that the two uncertainties $\Delta\phi$ and ΔL are related by $\Delta\phi\, \Delta L > \hbar$. Whereas for a stable state the uncertainty $\Delta\phi$ is arbitrarily large, this is different for a short-lived state because $\phi(t)$, being essentially proportional to the time t, must be limited in the same way as the lifetime is. A state with a finite mean life τ has its lifetime distributed according to a probability distribution

$(1/\tau)\ \exp(-t/\tau)$. The variance of the lifetime is therefore equal to

$$(\Delta t)^2 = [t^2] - [t]^2 = \tau^2 = \frac{\hbar^2}{\Gamma^2}.$$

If we use $\alpha_{\rm eff}$ as the effective constant of proportionality, so that

$$\phi(t) \approx \alpha_{\rm eff} t$$

then

$$\Delta\phi \approx \alpha_{\rm eff}\tau$$

and ΔL is bounded from below at approximately

$$\Delta L \approx \frac{\hbar}{\alpha_{\rm eff}\tau}.$$

The physical meaning of $\alpha_{\rm eff}$ is the angular velocity, which in the classical problem is related to the angular momentum and the kinetic energy by

$$L \approx M\alpha_{\rm eff} r^2$$

$$E_{\rm kin} \approx \alpha_{\rm eff}^2 r^2 M/2$$

where r is the sutiably averaged radius and M is the mass of the two body problem. Therefore $\alpha_{\rm eff}$ is of the order of $E_{\rm kin}/L$, and the lower bound for ΔL can also be written as

$$\Delta L \approx \frac{\hbar}{\tau}\frac{L}{2E_{\rm kin}}$$

$$\frac{\Delta L}{L} \approx \frac{\Gamma}{2E_{\rm kin}} \tag{24}$$

3.1.4. The Spin-1 W Boson, Away From its Mass Shell Can Assume Spin 0

Consider the decay of the spin-0 pion into a spin-0 lepton pair final state

$$\pi^+ \rightarrow \mu^+\nu$$

In the language of the Standard Model (Fig. 9) the quark–antiquark pair couples to a W^+ boson which couples in turn to the final state, which has spin 0.

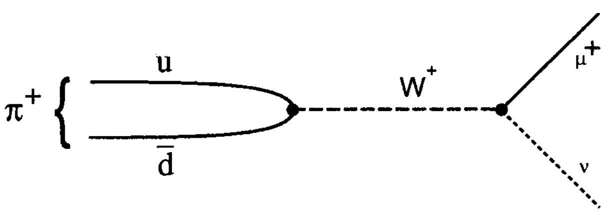

Fig. 9. Pion decay via an intermediate W-boson in the Standard Model.

The angular momentum at each vertex is conserved—the W^+ boson has spin 0 for the very short time of its existence which is of the order of

$$T \approx \frac{1}{M_W}$$

The natural lifetime of the W^+ is $\tau_W \gg T$. If during T the spin can be away from its on-shell value by 1 unit, one may expect it to be away during its natural lifetime by

$$\Delta S \approx \frac{T}{\tau_W} \approx \frac{\Gamma_W}{M_W}$$

3.2. Neighboring Spins and Suitable Representations

From a group theoretical point of view angular momentum eigenvalues are discrete numbers (integer or half-integer) because the underlying rotation group is compact. Therefore a non-zero spin spread can only mean the occupation of neighboring states in addition to the main state. How many neighboring states do we expect? For the moment we do not know and take the minimum that would describe the effect, i.e., in addition to the original J, we use $J+1$ and (if $J \geq 1$) $J-1$. Any spin characterization of short-lived particles should involve representations of the rotation group that have the required number of dimensions so that the total spin state can be accommodated as a vector in the representation space.

In the paper mentioned above (Blum and Saller, 2003) we have described relativistically compatible spin wave functions by embedding spinning particles into a Lorentz transformation compatible field. Here we can only briefly summarize some relevant results.

For stable states with integer spin, the general procedure starts with embedding the spin-0 particles with SU(2) representation D^0 in the Lorentz group representation $D^{[0|0]} \cong D^0$, then the spin-1 representation of the rotation group in the Lorentz group representation $D^{[\frac{1}{2}|\frac{1}{2}]} \cong D^1 \oplus D^0$.

For unstable states we chose the next higher Lorentz group representation so that D^0 is embedded in $D^{[\frac{1}{2}|\frac{1}{2}]} \cong D^1 \oplus D^0$ and D^1 in $D^{[1|1]} \cong D^2 \oplus D^1 \oplus D^0$. Whereas this embedding is not unique, our choice is for the two (left and right spin) indices to be "as close as possible to each other."

3.3. Constructing Wave Functions for Unstable States

3.3.1. Stable State, Spin Ignored

For comparison, a stable state with a given energy E is obtained from harmonic analysis of the propagator

$$\psi(t, r) = -\int \frac{d^4q}{2\pi^2} \frac{1}{q^2 + io - m^2} e^{-iqx} \delta(q_0 - E) = e^{-iEt} \frac{e^{ikr}}{r} \qquad (25)$$

Here $k = \sqrt{E^2 - m^2}$, the Dirac δ-distribution is used to pick the given energy from the sea of all possible energies. The outgoing spherical wave is obtained by using a suitable prescription of the integration path in the complex $|\vec{q}|$-plane.

3.3.2. *Unstable State, Spin Ignored*

For an unstable state, two modifications are introduced: Firstly, the distributional "io" in Eq. (25) is replaced by the non-zero invariant width "$im\Gamma$." Secondly, the effect of the correlations as in (15) must be taken into account. For the special case of the radial wave function ($\overrightarrow{\Delta k}$ and $\vec{k}$ both radially outward), Eq. (15) takes the form

$$\frac{E_i - E}{\Delta E} = \frac{k_i - k}{\Delta k} \tag{26}$$

In the integral (25) the Dirac δ-distribution $\delta(q_0 - E)$ is replaced by one which picks the correlation (26) instead of the fixed energy

$$\begin{aligned}\delta(q_0 - E) \to \delta\left(\frac{\Delta k}{\sqrt{(\Delta k)^2 - (\Delta E)^2}}(q_0 - E) - \frac{\Delta E}{\sqrt{(\Delta k)^2 - (\Delta E)^2}}(|\vec{q}| - k)\right)\\ \stackrel{\text{def}}{=} \delta(C(q_0 - E) - S(|\vec{q}| - k))\end{aligned} \tag{27}$$

The Lorentz-invariant square root guarantees that for $\Delta E = 0$ the two δ-distributions are the same.

For an unstable spherical outgoing wave, characterized by (E, k) and $(\Delta E, \Delta k)$ the wave function takes the form

$$\psi(t, r) = -\int \frac{d^4 q}{2\pi^2} \frac{1}{q^2 + im\Gamma - m^2}\, e^{-iqx} \delta(C(q_0 - E) - S(|\vec{q}| - k)), \tag{28}$$

which integrates (Blum and Saller, 2003) to

$$\psi(t, r) = e^{-i(E - i\Delta E)t}\, \frac{e^{i(k - i\Delta k)r}}{r} \tag{29}$$

in first-order of Γ/m.

3.3.3. *Stable State, Spin 0 With Neighbor*

The new degrees of freedom are introduced by choosing the Lorentz representation $D^{[\frac{1}{2}|\frac{1}{2}]} \cong D^1 \oplus D^0$. Starting from Eq. (25) we introduce the spin-Lorentz transmutator Λ_k^j applicable for the vector representations $D^{[\frac{1}{2}|\frac{1}{2}]}$, the vector representations of the boost

$$\Lambda\left(\frac{q}{m}\right)_k^j = \frac{1}{m}\begin{pmatrix} q_0 & \vec{q} \\ \vec{q} & \delta^{ab}m + \frac{q_a q_b}{q_0+m} \end{pmatrix} \tag{30}$$

The wave function has four components, one for the main spin ($j = 0$), and three for the spin neighbors ($j = 1, 2, 3$). The wave function is

$$A^j(t,r) = -\int \frac{d^4q}{2\pi^2}\Lambda\left(\frac{q}{m}\right)_0^j \frac{1}{q^2 + io - m^2}\, e^{-iqx}\delta(q_0 - E) \tag{31}$$

$$= \begin{pmatrix} \frac{i}{m}\partial_t \\ -\frac{\vec{x}}{r}\frac{i}{m}\partial_r \end{pmatrix} e^{-iEt}\,\frac{e^{ikr}}{r} = \begin{pmatrix} \frac{i}{m}\partial_t \\ -\frac{\vec{x}}{r}\frac{i}{m}\partial_r \end{pmatrix} e^{-iEt}\, kh_0^+(kr)$$

$$= e^{-iEt}\begin{pmatrix} \frac{E}{m}kh_0^+(kr) \\ \frac{\vec{x}}{r}i\frac{k}{m}kh_1^+(kr) \end{pmatrix} \tag{32}$$

Here the two Hankel functions $h_l^+(\rho)$ appear which are associated with angular momentum $l = 0$ and $l = 1$ (Messiah, 1959) $h_0^+(\rho) \equiv e^{i\rho}/\rho$ and $h_1^+(\rho) = -(d/d\rho)h_0^+(\rho)$. In the central rest system, Eq. (32) is

$$A^j(t,r) \stackrel{\text{rs}}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}\frac{1}{r} + \cdots \tag{33}$$

(We are interested in the leading term proportional $1/r$).

Let the spin function in front of the space–time-dependent part of the total wave function be called χ. It has gone from a scalar in Eq. (25) to a four-component vector in Eq. (33)

$$\chi_{\text{stable}}^{J=0}1 \to \chi_{\text{stable}}^{J=0+1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{34}$$

There is no new information in the additional three components.

3.3.4. Unstable State, Spin 0 With Neighbor

For this case we have to combine Eqs. (28) and (31). The wave function is

$$A_\Gamma^j(t,r) = -\int \frac{d^4q}{2\pi^2}\Lambda\left(\frac{q}{m}\right)_0^j \frac{1}{q^2 + im\Gamma - m^2}\, e^{-iqx}$$
$$\times\, \delta(C(q_0 - E) - S(|\vec{q}| - k)) \tag{35}$$
$$= \begin{pmatrix} \frac{i}{m}\partial_t \\ -\frac{\vec{x}}{r}\frac{i}{m}\partial_r \end{pmatrix} e^{-i(E-i\Delta E)t}\, \frac{e^{i(k-i\Delta k)r}}{r} \tag{36}$$
$$= \begin{pmatrix} \frac{E-i\Delta E}{m} k h_0^+(kr) \\ i\frac{\vec{x}}{r}\frac{k}{m}\left[k h_1^+(kr) - \Delta k h_0^+(kr)\right] \end{pmatrix} e^{-i(E-i\Delta E)t}\, e^{\Delta kr} \tag{37}$$

In comparison between Eqs. (37) and (32), new components proportional to ΔE and Δk have appeared in the unstable case. When going to the central rest frame ($E = m$, $k = 0$) this may be expressed by the spin function as

$$\chi^{J=0+1}_{\text{unstable}} = \chi^{J=0+1}_{\text{stable}} + \begin{pmatrix} -i\frac{\Gamma}{2m} \\ -i\frac{\vec{x}}{r}\frac{\Delta k^*}{m} \end{pmatrix} \tag{38}$$

As the spin-0 particle has become unstable, the spin fucntion has acquired a spin-1 component in the new degrees of freedom, proportional to Δk^*.

3.3.5. Spin Spread as R.M.S. Variation Over Discrete Spins

To have a quantitative measure of the variation of the discrete spin we define a "spin spread" ΔS as the root-mean-square variation of the spin intensity of the short-lived state. The intensity in the new spin-1 component in (38), mixed into the original spin-0 state is equal to

$$\left(\frac{\Delta k^*}{m}\right)^2,$$

whereas the intensity in the original spin-0 state is 1 (Γ^2/m^2 neglected against 1). To first-order of Γ/m we find for the case at hand

$$\Delta S = \frac{|\Delta k^*|}{m} \geq \frac{\Gamma}{2m} \tag{39}$$

3.4. Comparison With the Conventional Treatment of Spin for Resonances

Traditionally there is only one discrete number (integer or half-integer) for the spin of a resonance, and there is no spin spread, even if the resonance is quite wide, i.e. even if it has a fairly large value of Γ/m.

3.4.1. Partial Wave Method in Formation

In a formation experiment with hadrons, Section 2.1, one would measure the distribution of the scattering angle θ as a function of the incoming momentum k. One would analyze the angular distribution using the method of partial waves (Bohm, 1993; Roman, 1965) as follows.

The amplitude $f(\theta)$ for elastic scattering in the center of mass system is decomposed into partial waves of angular momentum l by the series

$$f(\theta) = \frac{1}{p}\sum_{l=0}^{\infty}(2l+1)\,\frac{e^{2i\delta_l}-1}{2i}\,P_l(\cos\theta) \tag{40}$$

where $P_l(\cos\theta)$ is the Legendre polynomial of degree l and p the c.m. momentum uniquely related to the total energy E^{cm} in the c.m., δ_l is the phase shift of the partial wave l and depends on one variable, say p or E^{cm}. The differential cross section and with it the angular distribution are given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2. \tag{41}$$

It is possible to experimentally determine the δ_l's as functions of E^{cm} up to some $l_{\max}$ $(l = 0, 1, \ldots l_{\max})$ which is given by the finite range of forces between the hadrons. (The partial wave method can be generalized to the case of inelastic scattering by introducing complex phase shifts $\eta_l = \delta_l + i\gamma_l$ where the γ's describe the degree of inelaticity in the angular momentum l (Roman, 1965). The essential element in the traditional method of giving a resonance only one discrete number for angular momentum can be discussed with real δ_l's.)

Assuming that background plays no important rôle, the intermediate state of Fig. 1 is identified as being a resonance in the partial wave l when one of the $\delta_l\,(E^{\mathrm{cm}})$ passes through the value of 90° at some $E^{\mathrm{cm}} = E^{\mathrm{cm}}_{\mathrm{res}}$. The width of the resonance is related to the speed $d\delta_l/dE^{\mathrm{cm}}$ with which the phase passes through 90° as the center of mass energy is varied. This interpretation of the resonance phenomenon is in the direct line of a development from mechanical or electrodynamical forced oscillators with damping, over optical resonances, to resonances occurring in potential scattering of the Schrödinger theory. The method is well established also in the identification of the angular momentum of nuclear states. If it is battle tested that well—then what is the problem?

The problem comes in two steps: (1) A relativistic description of a resonance requires that the phase is a function not only of the energy but also of the momentum of the resonance. (2) Traditionally it was possible to eliminate the momentum as a second variable by going to the rest frame where it was zero; this is no longer possible due to the momentum uncertainty Δk^* which is present in the central rest frame, and which is at least as large as $\Gamma/2$. In fact, a partial wave resonance interpretation implicitly assumes Eq. (18) to be valid.

3.4.2. Decay Angular Distribution Method in Production

A proper description of the spin of a resonance is intimately related to our insistence that the resonance must be describable in any one but only one chosen Lorentz frame.

This is also elucidated in the following situation. Imagine a ρ^0-resonance ($m = 770$ MeV, $\Gamma = 150$ MeV) which is known to predominantly decay into $\pi^+\pi^-$. Let it be produced in well-defined conditions in a pure state (e.g., in $\pi^- p \to \rho^0 n$ ($\rho^0 \to \pi^+\pi^-$) at a fixed laboratory momentum of the incoming π^- on a stationary polarized target, in the forward direction). With respect to its mass, the ρ^0-meson consists of the ensemble of all the possibilities to manifest itself with some mass value distributed around the central value m.

In the central rest system of the ρ^0, the two decay particles of the central mass are back-to-back; let the direction of the π^+ be called $\hat{p}$. A pure spin-1 property of the ρ-meson requires that the angular wave function $f(\hat{p})$ be a vector representation of the rotation group, e.g. a linear combination of the spherical harmonics for $l = 1$.

$$f(\hat{p}) = \alpha_{-1} Y_1^{-1}(\hat{p}) + \alpha_0 Y_1^0(\hat{p}) + \alpha_{+1} Y_1^{+1}(\hat{p}) \tag{42}$$

Staying in the same central rest system, the decay pattern is no longer back-to-back at mass values different from the central mass m, because the additional momentum present in the central rest frame is added to the momenta of the decay particles. The original direction $\hat{p}$ is shifted the more one goes away from m (cf. Fig. 10).

$$\hat{p} \to \hat{p}'.$$

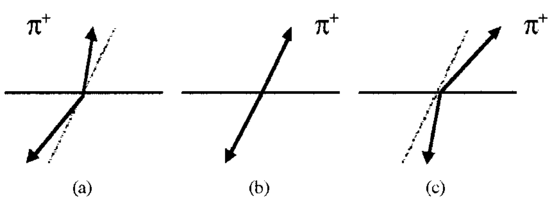

Fig. 10. Decay patterns of $\rho \to \pi^+\pi^-$ belonging to the ensemble of masses, measured in the central rest frame, (a) at $m - \Gamma/2$, (b) at m, (c) at $m + \Gamma/2$. The momentum uncertainty Δk^* is thought to be along the horizontal line.

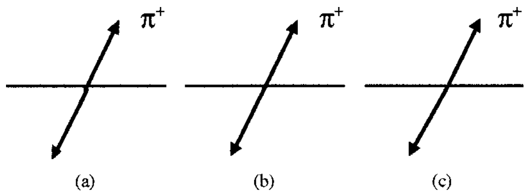

Fig. 11. Decay patterns of $\rho \to \pi^+\pi^-$ belonging to the ensemble of masses, measured in three different Lorentz frames: (a) the one at $m - \Gamma/2$, (b) the one at m, (c) the one at $m + \Gamma/2$; they are identical.

Directions $\hat{p}'$ no longer follow a relation like (42), and other spherical harmonics ($l \neq 1$) come in, thus producing spin neighbors.

Traditionally one did not stay in the same system but went to a different Lorentz frame for every mass value of the ensemble, thus causing the decay products to stay back-to-back, and keeping the relation (42) valid over the entire mass range of the resonance (see Fig. 11). The Lorentz frame was selected event by event, according to which value of the mass was accidentally realized. This procedure was of course not "wrong," but it required for the description of the resonance a continuous family of Lorentz frames, and it did not allow to describe the resonance in one unique Lorentz frame.

4. CONCLUSIONS

We have shown that a relativistically compatible definition of unstable states is possible without referring to constituents or the dynamical behavior of decay products. Questions as to the mathematical foundations remain open. (e.g., the conventional Hilbert space is not a suitable framework as it does not allow for decaying states (Bohm, 1993).

The appearance of non-zero uncertainties (spreads) of energy, momentum and spin requires no complex boost parameter for the Lorentz group; this is in harmony with the procedure of Bohm and Kaldass (1999). On the other hand, the non-zero $\overrightarrow{\Delta k}^*$, the momentum uncertainty in the central rest frame of the decaying particle, appears to be the key to an understanding of the unstable states. It is the reason for the existence of two species of Lorentz frames, the central rest frames ($\vec{k} = 0$) and the sharp-energy systems ($\Delta E = 0$).

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